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# Transverse instability of solitary-wave states of the water-wave problem

By T. J. BRIDGES

Department of Mathematics and Statistics, University of Surrey, Guildford, Surrey GU2 7XH

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Transverse stability and instability of solitary waves correspond to a class of perturbations that are travelling in a direction transverse to the direction of the basic solitary wave. In this paper we consider the problem of transverse instability of solitary waves for the water-wave problem, from both the model equation point of view and the full water-wave equations. A new universal geometric condition for transverse instability forms the backbone of the analysis. The theory is first illustrated by application to model PDEs for water waves such as the KP equation, and then it is applied to the full water-wave problem. This is the first theory proposed for transverse instability of solitary waves of the full water-wave problem. The theory suggests the introduction of a new functional for water waves, whose importance is suggested by the mathematical structure. Without explicit calculation, the theory is used to argue that the basic class of solitary waves of the water-wave problem, which bifurcate at Froude number unity, are likely to be stable to transverse perturbations, even at large amplitude.

## 1. Introduction

Given a solitary wave travelling uniformly in one space direction, a *transverse instability* of the solitary wave is an instability associated with a class of perturbations that are travelling in a direction transverse to the basic direction.

The problem of transverse instability of solitary waves was first considered by Kadomstev & Petviashvili (1970), and the model equation they constructed to address the question of the transverse instability of the Korteweg–de Vries (KdV) solitary wave is the celebrated KP equation

$$u_t + uu_x + u_{xxx} = v_y \quad \text{and} \quad v_x + \sigma u_y = 0, \quad \sigma = \pm 1. \tag{1.1}$$

Kadomstev & Petviashvili (1970) studied the stability of the KdV solitary wave to transverse perturbations by linearizing (1.1) about it and approximating the stability exponents. They found that the KdV solitary wave is transverse unstable when $\sigma = -1$ and transverse stable when $\sigma = +1$. Shortly thereafter it was recognized that the KP equation is a completely integrable Hamiltonian partial differential equation. Therefore the transverse instability result for KP and other integrable models could be deduced a number of different ways, including explicit calculation (cf. Zakharov 1975; Makhankov 1978; Kuznetsov, Rubenchik & Zakharov 1986; Infeld & Rowlands 1990; Alexander, Pego & Sachs 1997; Allen & Rowlands 1997).

Since the KP results, transverse instability of solitary-wave states of several model PDEs have been studied. The most well-studied example is the nonlinear Schrödinger (NLS) equation (cf. Makhankov 1978; Laedke & Spatschek 1978; Janssen & Rasmussen 1983; Rypdal & Rasmussen 1989); see Kivshar & Pelinovsky (2000) for a recent review of transverse instability for NLS and related models including the Davey–Stewartson equation. The Zakharov–Kuznetsov equation, which is another generalization of the KdV equation to higher space dimension, has also been studied for transverse instability (cf. Zakharov & Kuznetsov 1974; Spatschek, Shukla & Yu 1975; Iwasaki, Toh & Kawahara 1990; Allen & Rowlands 1993). Methods which have been used to study the linearized stability equation associated with transverse instability include analytical techniques based on integrable models (cf. Kuznetsov *et al.* 1986 for a review), theories based on modulation equations (cf. Kadomstev & Petviashvili 1970; Ostrovsky & Shrira 1976; Shrira 1980; Janssen & Rusmussen 1983; Shrira & Pesenson 1984), variational approximations such as the Rayleigh–Ritz method (cf. Laedke & Spatschek 1978; Allen & Rowlands 1993, 1995; Bettinson & Rowlands 1997) and direct numerics (cf. Iwasaki *et al.* 1990; Infeld, Rowlands & Senatorski 1999).

In order to distinguish between transverse instabilities and instabilities travelling in the same direction as the basic state, the latter will be referred to throughout as *longitudinal instabilities*.

While model equations for the water-wave problem such as the KP equation have been studied for transverse instability, the transverse instability question for the full water wave problem has never been studied. Indeed, as far as we are aware, there are only three papers on the longitudinal instability – where the class of perturbations travels in the same direction as the basic wave – of solitary waves of the full water wave problem: Tanaka (1986), Tanaka *et al.* (1987) and Longuet-Higgins & Tanaka (1997). In fact there is a close connection between all three of these papers and Saffman's theory for the superharmonic instability of periodic travelling waves.

The superharmonic instability of *periodic travelling waves* corresponds to an instability where the perturbation has the same wavelength as the basic travelling wave (this is in contrast to the Benjamin–Feir instability or the subharmonic instability where the wavelength of the perturbation differs from the basic wavelength). For gravity waves, superharmonic instability occurs at very large amplitude and was first discovered numerically (Longuet-Higgins 1978), and later Tanaka (1985) found numerically that an exchange of stability occurs at precisely the value of the wave speed $c$ where $\mathrm{d}I/\mathrm{d}c$ first changes sign, where $I$ is the impulse of the solitary wave. Saffman (1985) then proved that when $\mathrm{d}I/\mathrm{d}c$ changes sign, an eigenvalue of the linear stability problem changes from stable to unstable (or vice versa), generically. An important implication of this result is that information about stability can be deduced by plotting the impulse $I$ against the wave speed $c$ for a family of periodic travelling waves.

The main result of Tanaka (1986) was to show numerically that a similar result was true for gravity solitary waves. By calculating the eigenvalues of the linearization of the full water-wave problem about finite-amplitude solitary gravity waves, he was able to show that there was a change from stability to instability at precisely the point where the slope of the impulse $I$ of the solitary wave plotted against the wave speed $c$ first changed sign. For solitary gravity waves, the plot of $I$ against $c$ when continued beyond the first change of slope has additional points where $\mathrm{d}I/\mathrm{d}c$ changes sign. A qualitative description of the impulse plotted as a function of the wave speed for gravity solitary waves is shown in figure 1.

This connection between the sign of $\mathrm{d}I/\mathrm{d}c$ and instability motivated Longuet-Higgins & Tanaka (1997) to study the linear stability problem in more detail to see if additional unstable eigenvalues in the linearization about a solitary wave are generated by these sign changes. Indeed, they computed eigenvalues associated with

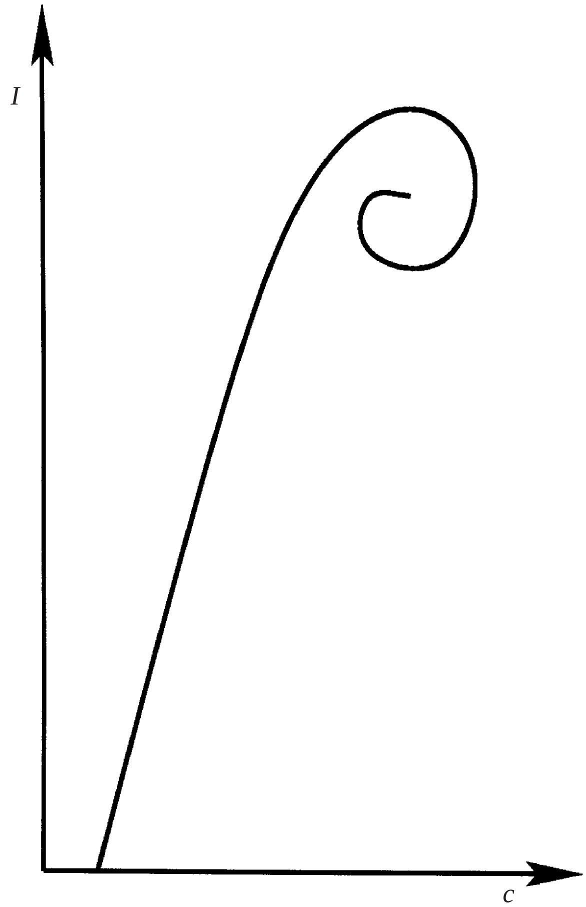


FIGURE 1. Schematic of the impulse $I$ versus wave speed $c$ for gravity solitary waves.

the first two changes of sign of $\mathrm{d}I/\mathrm{d}c$ and confirmed that unstable eigenvalues were generated at each point, and they conjectured that further changes in sign of $\mathrm{d}I/\mathrm{d}c$ would generate more unstable eigenvalues.

These longitudinal instabilities are often referred to as *crest instabilities*. In a remarkable paper following up on the numerical observations of Tanaka (1986), Tanaka *et al.* (1987) showed a connection between this type of instability of the solitary wave and a form of wave breaking. This connection was established by integrating the initial-value problem for water waves with initial data chosen to be an eigenfunction associated with an unstable solitary wave. Although not all initial data evolved to wave breaking, in cases where wave breaking occurred the connection with instability was clearly established. A representative example of this is shown in figure 5 of Tanaka *et al.* (1987). This observation is of interest here because we find that transverse instability also occurs when $\mathrm{d}I/\mathrm{d}c$ has a negative sign, and therefore it is reasonable to conjecture that the generated *transverse crest instability* may lead to a form of wave breaking with transverse modulation.

The connection between the sign of $\mathrm{d}I/\mathrm{d}c$ and instability can be deduced from the Hamiltonian structure alone. For periodic waves, this is precisely the argument used by Saffman (1985). It is associated with the fact that such states are associated with a symmetry and can be classified as relative equilibria, and there are established results in Hamiltonian dynamics for such states. For solitary waves of model equations, there are now many results in the literature on the sign of $\mathrm{d}I/\mathrm{d}c$ and the longitudinal instability of solitary waves (cf. Grillakis, Shatah & Strauss 1990; Pelinovksy & Grimshaw 1997, and references therein). As a way of leading into the discussion on transverse instability using multi-symplectic structure, we point out that – even for longitudinal instabilities of solitary waves – much more information about the connection between $\mathrm{d}I/\mathrm{d}c$ and instability can be obtained by multi-symplectifying the governing equations. For example, in Bridges & Derks (1999, 2000), it is shown that it is possible to deduce which sign of $\mathrm{d}I/\mathrm{d}c$ corresponds to instability, without information about the spectral problem, and it can be shown that $\mathrm{d}I/\mathrm{d}c$ must be corrected by another term – which occurs naturally in the multi-symplectic formulation – when the solitary wave has oscillatory or other non-trivial tails.

In this paper, the transverse instability problem for the full water-wave problem is formulated and studied for the first time. The backbone of the analysis is a new universal instability condition for transverse instability (Bridges 2000). This condition is universal in the sense that it can be derived independent of a particular equation. It is a property of the multi-symplectic structure of the equations, and applies to solitary states of any system of PDEs which has this structure.

Another way to view this transverse instability condition is as a generalization of the above idea where changes of sign of $\mathrm{d}I/\mathrm{d}c$, when $I(c)$ is evaluated on a solitary wave, are associated with the generation of unstable eigenvalues in the linearization about the solitary wave. We will introduce a pair of functions $\mathscr{A}(c,\ell)$ and $\mathscr{B}(c,\ell)$ where $\mathscr{A}(c,\ell)$ has the property that

$$\lim_{\ell\to 0}\mathscr{A}(c,\ell) = -I(c). \tag{1.2}$$

The function $\mathscr{A}(c,\ell)$ is a one-parameter extension of the impulse. The function $\mathscr{B}(c,\ell)$ is new, and it is the function that carries information about transverse instability. For the water-wave problem, with velocity potential $\phi$ and wave height $\eta$, we will show in § 6 that it is given explicitly by

$$\mathscr{B}(c,\ell) = \ell\int_{-\infty}^{+\infty}\left[\frac{\tau\hat{\eta}_\theta^2}{\sqrt{1+(1+\ell^2)\hat{\eta}_\theta^2}} + \int_0^{\hat{\eta}}(\hat{\phi}_\theta)^2\,\mathrm{d}z\right]\mathrm{d}\theta, \tag{1.3}$$

where the hats indicate that $\hat{\eta}$ and $\hat{\phi}$ are evaluated on a solitary wave, $\tau$ is the coefficient of surface tension and $\theta = x - ct + \ell y + \theta_0$. The second parameter here, $\ell$, represents a rotation of the uni-directional travelling solitary wave away from the $x$-axis.

There is no apparent physical interpretation of the functional $\mathscr{B}(c,\ell)$. As far as we are aware the importance of this function for solitary waves – or water waves in general – has not been previously recognized, and it is not obvious that this function carries information about the transverse instability of solitary waves. It arises naturally in the derivation in § 6 from the transverse component of the multi-symplectic structure. It is an example of a functional whose importance has been dictated by the mathematical structure of the equations rather than physical considerations.

Using the above two functions, we will show the following: if

$$\det\begin{bmatrix}\mathscr{A}_c & \mathscr{A}_\ell\\ \mathscr{B}_c & \mathscr{B}_\ell\end{bmatrix} > 0, \tag{1.4}$$

when evaluated on a solitary wave, it is transverse unstable. This result is a special case, applied to water waves, of the result in Bridges (2000). We will show that it is a *sufficient condition for transverse instability*, and the converse is a necessary but not sufficient condition for transverse stability.

An even more remarkable result – which does not hold in general, but does apply to water waves – is the following. Suppose the solitary wave is restricted to travel in a direction parallel to the $x$-axis (i.e. $\ell = 0$) and let

$$\hat{\mathscr{B}}(c) = \lim_{\ell \to 0} \frac{1}{\ell} \mathscr{B}(c, \ell). \tag{1.5}$$

Then, defining $\mathscr{A}(c, 0) = -I(c)$, we will show, for any solitary water wave for which the expressions $\hat{\mathscr{B}}(c)$ and $\mathrm{d}I/\mathrm{d}c$ exist, the solitary wave is *transverse* unstable if

$$\hat{\mathscr{B}}(c) \frac{\mathrm{d}I}{\mathrm{d}c} < 0. \tag{1.6}$$

This result uses information from the strictly one-dimensional problem (i.e. without rotating the wave into the transverse direction) to deduce instability information in the transverse direction. This result is particularly useful for water waves because $\hat{\mathscr{B}}(c)$ is strictly positive. The simplification of (1.4) to (1.6) does not occur in all systems. It is shown in § 5 that the simplification (1.6) arises only in systems with a transverse reflection symmetry, which is present in water waves.

Using numerical results of Longuet-Higgins (1974) and Tanaka (1986) on $I(c)$, which are qualitatively represented by figure 1, it is apparent that $\mathrm{d}I/\mathrm{d}c > 0$ for gravity solitary waves for most of the branch of waves, with the first change of sign occurring at large amplitude. The above theory suggests that these waves are quite robust against transverse perturbations – until $\mathrm{d}I/\mathrm{d}c$ becomes negative. However, this statement must be qualified, because 'stability' is only with respect to transverse perturbations with large spanwise wavenumber (see § 7 for precise specification of the class of perturbations).

The result (1.6) also applies to capillary–gravity solitary waves – i.e. whenever $\mathrm{d}I/\mathrm{d}c < 0$ a capillary–gravity solitary wave is transverse unstable. However, there is very little information available in the literature about the value of the momentum along branches of capillary–gravity solitary waves.

The key to the instability theory is a formulation of Hamiltonian PDEs as multi-symplectic systems. Multi-symplecticity is a generalization of Hamiltonian structure where a distinct symplectic structure is assigned for time and each space direction (cf. Bridges [1996, 1997*a*, *b*, 1998, 1999]). Given a classical Hamiltonian PDE it is straightforward to reformulate it as a multi-symplectic system. The function $\mathscr{B}(c, \ell)$ is then deduced precisely from the transverse symplectic structure. The framework is relatively straightforward to set up, and – more importantly – the results obtained in the multi-symplectic setting can be translated back into the original physical coordinates for combination with analysis or numerics of water waves or other Hamiltonian PDEs with solitary wave states.

In § 2 we take as a starting point a Hamiltonian PDE formulated as a multi-symplectic system, and show that this form leads to a new constrained variational principle for solitary waves travelling in one dimension, but dependent on two parameters $(c, \ell)$ rather than just $c$. In § 3 we introduce the new geometric instability condition for transverse instability first presented in Bridges (2000), and to illustrate it, it is applied to transverse instability of the KP model.

In § 4, a detailed example of how the theory is applied to a particular problem is given, using KP as an example. Starting with the PDE in its classical form, it is first reformulated as a multi-symplectic system. The structure then generates the required functionals $\mathscr{A}(c,\ell)$ and $\mathscr{B}(c,\ell)$, which are then evaluated on a two-parameter family of solitary waves.

In § 5 the implications of a transverse reflection symmetry – which appears in many PDEs including KP and the water-wave problem – is considered and we show that it leads directly to the intriguing result (1.6). In § 6 the multi-symplectic form of the water-wave problem is given and the properties of the transverse instability condition studied. The theory shows that $\mathscr{B}(c,\ell)$ defined in (1.3) above is indeed the relevant function which encodes information about transverse instability of oceanographic solitary waves. In § 7 the stability problem is formulated, and it is shown that when (1.4) is satisfied for a basic solitary wave, there exists an unstable eigenvalue of the linearized stability problem whose growth rate has magnitude of order $\beta$, where $\beta$ is the wavenumber in the transverse direction of unstable eigenfunction. The result shows that there is a non-zero angle between the direction of travel of the unstable eigenfunction and the direction of travel of the basic state.

The theory should apply to a large range of other examples in ocean dynamics, internal wave dynamics, atmospheric dynamics, and with suitable generalization to NLS type models in optical fibre transmission, as well as other Hamiltonian PDEs. In § 8 we present a sample of other intriguing systems related to water waves to which the theory presented here can be applied. In the Appendix, some technical details needed for transforming the multi-symplectic structure for water waves into a standard form are recorded.

## 2. Multi-parameter families of solitary waves and transverse symplecticity

The starting point for the analysis is the following abstract formulation of Hamiltonian partial differential equations:

$$\mathbf{M}Z_t + \mathbf{K}Z_x + \mathbf{L}Z_y = \nabla S(\boldsymbol{Z}), \quad \boldsymbol{Z} \in \mathbb{X}, \tag{2.1}$$

where $\mathbb{X}$ is the *phase space*. For example, $\mathbb{X} = \mathbb{R}^8$ for the KP equation (see § 4) and for the water-wave problem it is a space of functions dependent on one variable over a finite interval (the vertical direction).

The operators $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{L}$ are constant skew-symmetric operators, and $\nabla S(\boldsymbol{Z})$ is the gradient of a function $S : \mathbb{X} \to \mathbb{R}$ with respect to an inner product on $\mathbb{X}$, which will be denoted by $\langle \cdot, \cdot \rangle$ throughout, and $S(\boldsymbol{Z})$ is normalized so that $S(0) = 0$. The three skew-symmetric operators $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{L}$ define closed two forms by

$$\omega^{(1)}(\boldsymbol{U}, \boldsymbol{V}) = \langle \mathbf{M}\boldsymbol{U}, \boldsymbol{V} \rangle, \quad \omega^{(2)}(\boldsymbol{U}, \boldsymbol{V}) = \langle \mathbf{K}\boldsymbol{U}, \boldsymbol{V} \rangle \quad \text{and} \quad \omega^{(3)}(\boldsymbol{U}, \boldsymbol{V}) = \langle \mathbf{L}\boldsymbol{U}, \boldsymbol{V} \rangle, \tag{2.2}$$

where $\boldsymbol{U}$ and $\boldsymbol{V}$ are any vectors in $\mathbb{X}$. We will be predominantly interested in solitary waves that are travelling in the $x$-direction or at small angles to the $x$-axis. Therefore we refer to the symplectic structure associated with $\omega^{(3)}$ as the *transverse symplectic structure*.

The theory of transverse instability to be presented here is independent of any particular PDE, it depends only on the abstract form (2.1). Therefore the theory will apply to any system which can be cast into the form (2.1). It will be assumed that a uni-directional travelling solitary wave exists and satisfies (2.1). The theory does not require an explicit form for this wave – only that it exists, is differentiable and decays exponentially to zero. The basic solitary wave is taken to be of the form

$$\boldsymbol{Z}(x, y, t) = \hat{\boldsymbol{Z}}(\theta; c, \ell) \quad \text{with} \quad \theta = x - ct + \ell y + \theta_0. \tag{2.3}$$

The parameters $c$ and $\ell$ represent the speed and transverse wavenumber respectively, and $\theta_0$ is an arbitrary real parameter.

The function $\hat{\boldsymbol{Z}}$ is required to satisfy the asymptotic boundary conditions

$$\lim_{\theta\to\pm\infty} \|\hat{\boldsymbol{Z}}(\theta;c,\ell)\| = 0, \tag{2.4}$$

where $\|\cdot\|$ is a norm on $\mathbb{X}$. Moreover the decay as $\theta \to \pm\infty$ is assumed to be exponential.

Substitution of this form into (2.1) leads to the following differential equation for $\hat{\boldsymbol{Z}}$:

$$\mathbf{J}(c,\ell)\hat{Z}_\theta = \nabla S(\hat{\boldsymbol{Z}}), \quad \text{where } \mathbf{J}(c,\ell) = \mathbf{K} - c\mathbf{M} + \ell\mathbf{L}. \tag{2.5}$$

The matrix $\mathbf{J}(c,\ell)$ is skew symmetric and defines the two form

$$\Omega(\boldsymbol{U},\boldsymbol{V}) = \langle \mathbf{J}(c,\ell)\boldsymbol{U},\boldsymbol{V}\rangle. \tag{2.6}$$

The system (2.5) is in standard form for a classical Hamiltonian system with pre-symplectic structure $\Omega$, but with evolution in the $\theta$-direction. When the system (2.5) reduces to an ODE, for example when $\mathbb{X}$ is finite-dimensional, and $\mathbf{J}(c,\ell)$ is invertible, the basic solitary wave can be characterized as a homoclinic orbit of a finite-dimensional Hamiltonian system.

Another view of the basic family of solitary waves, which will be important in the stability analysis, is as a solution of a constrained variational principle. Define

$$\mathscr{H}(\hat{\boldsymbol{Z}}) = \int_{-\infty}^{+\infty} [S(\hat{\boldsymbol{Z}}) - \tfrac{1}{2}\langle \mathbf{K}\hat{Z}_\theta, \hat{\boldsymbol{Z}}\rangle]\,\mathrm{d}\theta, \tag{2.7}$$

$$\mathscr{A}(\hat{\boldsymbol{Z}}) = -\int_{-\infty}^{+\infty} \tfrac{1}{2}\langle \mathbf{M}\hat{Z}_\theta, \hat{\boldsymbol{Z}}\rangle\,\mathrm{d}\theta = -\int_{-\infty}^{+\infty} \tfrac{1}{2}\omega^{(1)}(\hat{Z}_\theta, \hat{\boldsymbol{Z}})\,\mathrm{d}\theta, \tag{2.8}$$

$$\mathscr{B}(\hat{\boldsymbol{Z}}) = \int_{-\infty}^{+\infty} \tfrac{1}{2}\langle \mathbf{L}\hat{Z}_\theta, \hat{\boldsymbol{Z}}\rangle\,\mathrm{d}\theta = \int_{-\infty}^{+\infty} \tfrac{1}{2}\omega^{(3)}(\hat{Z}_\theta, \hat{\boldsymbol{Z}})\,\mathrm{d}\theta. \tag{2.9}$$

With the hypothesis (2.4), the exponential decay and differentiability of $\hat{\boldsymbol{Z}}$, and the normalization $S(0) = 0$, these integrals will exist.

The functional $\mathscr{H}(\hat{\boldsymbol{Z}})$ is the classical Hamiltonian function for the one-space-dimensional problem. The classical Hamiltonian function for the two-space-dimensional problem is in fact $\mathscr{H} - \ell\mathscr{B}$. Therefore, $\mathscr{H}$ and $\mathscr{B}$ can be thought of as giving a decomposition of the two-dimensional Hamiltonian function into a longitudinal part and a transverse part.

Solitary waves of the form (2.3)–(2.4) can then be formally characterized as critical points of the functional $\mathscr{H}(\hat{\boldsymbol{Z}})$ restricted to level sets of the two functionals $\mathscr{A}(\hat{\boldsymbol{Z}})$ and $\mathscr{B}(\hat{\boldsymbol{Z}})$, with $c$ and $\ell$ as Lagrange multipliers. The Lagrange functional is

$$\mathscr{F}(\hat{\boldsymbol{Z}},c,\ell) = \mathscr{H}(\hat{\boldsymbol{Z}}) - c\mathscr{A}(\hat{\boldsymbol{Z}}) - \ell\mathscr{B}(\hat{\boldsymbol{Z}}), \tag{2.10}$$

and it can be formally differentiated to obtain

$$\nabla\mathscr{F}(\hat{\boldsymbol{Z}},c,\ell) = \nabla\mathscr{S}(\hat{\boldsymbol{Z}}) - \mathbf{K}\hat{Z}_\theta + c\mathbf{M}\hat{Z}_\theta - \ell\mathbf{L}\hat{Z}_\theta = \nabla\mathscr{S}(\hat{\boldsymbol{Z}}) - \mathbf{J}(c,\ell)\hat{Z}_\theta;$$

that is, $\nabla\mathscr{F}(\hat{\boldsymbol{Z}},c,\ell) = 0$ formally recovers (2.5). Let $I_1$ and $I_2$ be the values of the constraint sets. Then the constrained variational principle is said to be non-degenerate when

$$\det\begin{bmatrix} \partial c/\partial I_1 & \partial c/\partial I_2 \\ \partial \ell/\partial I_1 & \partial \ell/\partial I_2 \end{bmatrix} \neq 0. \tag{2.11}$$

It follows from Lagrange multiplier theory that $c = \partial\mathscr{H}/\partial I_1$ and $\ell = \partial\mathscr{H}/\partial I_2$, in which case (2.11) can be written

$$\det \mathrm{Hess}_I(\mathscr{H}) = \det \begin{pmatrix} \partial^2\mathscr{H}/\partial I_1^2 & \partial^2\mathscr{H}/\partial I_1\partial I_2 \\ \partial^2\mathscr{H}/\partial I_2\partial I_1 & \partial^2\mathscr{H}/\partial I_2^2 \end{pmatrix} \neq 0. \tag{2.12}$$

In general, it is not expected that solitary waves will be maxima or minima of this constrained variational principle. In fact, for most interesting examples, the functionals involved are strongly indefinite. However, we will show that – regardless of the critical point type of the solitary wave – the parameter structure associated with the constrained critical point characterization of the solitary wave plays an important part in the transverse stability analysis.

Dual to the matrix in (2.11) is the Jacobian

$$\begin{pmatrix} \mathscr{A}_c & \mathscr{A}_\ell \\ \mathscr{B}_c & \mathscr{B}_\ell \end{pmatrix} = \begin{pmatrix} \partial c/\partial I_1 & \partial c/\partial I_2 \\ \partial \ell/\partial I_1 & \partial \ell/\partial I_2 \end{pmatrix}^{-1}. \tag{2.13}$$

Assuming the function describing the solitary wave $\hat{\boldsymbol{Z}}(\theta; c, \ell)$ is sufficiently smooth, this matrix can be written in the following interesting form, by formally differentiating the expressions in (2.8)–(2.9), using an integration by parts and the hypothesis (2.4):

$$\begin{pmatrix} \mathscr{A}_c & \mathscr{A}_\ell \\ \mathscr{B}_c & \mathscr{B}_\ell \end{pmatrix} = \int_{-\infty}^{+\infty} \begin{pmatrix} -\omega^{(1)}(\hat{Z}_\theta, \hat{Z}_c) & -\omega^{(1)}(\hat{Z}_\theta, \hat{Z}_\ell) \\ +\omega^{(3)}(\hat{Z}_\theta, \hat{Z}_c) & +\omega^{(3)}(\hat{Z}_\theta, \hat{Z}_\ell) \end{pmatrix} \mathrm{d}\theta. \tag{2.14}$$

The importance of this formula is that it connects the $(c, \ell)$ parameter structure of the two-parameter family of solitary waves with geometric properties of the equation, namely the symplectic structures associated with the time direction ($\omega^{(1)}$) and transverse direction ($\omega^{(3)}$). It will be established in § 7 that these two distinct symplectic structures encode geometric information about the *temporal* instability associated with the *transverse* direction.

## 3. A universal geometric condition for transverse instability

The main result needed for the study of transverse instability is the following geometric condition. *Suppose*

$$\det \begin{bmatrix} \mathscr{A}_c & \mathscr{A}_\ell \\ \mathscr{B}_c & \mathscr{B}_\ell \end{bmatrix} > 0 \tag{3.1}$$

*then the basic solitary wave is linearly transverse unstable.* This condition was first presented in Bridges (2000) and the details of the argument from that paper needed here are given in § 7.

To illustrate how this result may be applied, consider the following generalization of the KP equation:

$$(2u_t + f(u)_x + u_{xxx})_x + \sigma u_{yy} = 0, \tag{3.2}$$

where $\sigma = \pm 1$, $f(u)$ can be any smooth function, and the 2 multiplying $u_t$ is added for convenience. Let

$$f(u) = \frac{u^{p+1}}{p+1}, \tag{3.3}$$

then this form of the KP equation has an oblique uni-directional travelling solitary-wave solution of the form

$$u(x, y, t) = \hat{u}(\theta; c, \ell) = a(c, \ell)\,\mathrm{sech}^{2/p}(\gamma(c, \ell)\theta), \tag{3.4}$$

where $\theta = x - ct + \ell y + \theta_0$,

$$\gamma(c, \ell) = \tfrac{1}{2} p b(c, \ell), \quad a(c, \ell) = [\tfrac{1}{2} b(c, \ell)^2 (p+1)(p+2)]^{1/p}, \quad b(c, l) = [2c - \sigma\ell^2]^{1/2}, \tag{3.5}$$

with the assumption that $2c - \sigma\ell^2 > 0$.

In order to apply the instablity criterion (3.1) we need the two functions $\mathscr{A}(c, \ell)$ and $\mathscr{B}(c, \ell)$. The functional $\mathscr{A}(c, \ell)$ is just an extended (i.e. when $\ell \neq 0$) form of the momentum or impulse for the KP model which is

$$\mathscr{A}(c, \ell) = -\int_{-\infty}^{+\infty} (\hat{u}(\theta; c, \ell))^2 \,\mathrm{d}\theta. \tag{3.6}$$

(We will also deduce this expression from the multi-symplectic structure in § 4.) However the form of the functional $\mathscr{B}(c, \ell)$ is not obvious and only becomes apparent from the transverse symplectic structure. The transverse symplectic structure for the KP model is constructed in § 4. The resulting form for $\mathscr{B}(c, \ell)$ is found to be

$$\mathscr{B}(c, \ell) = \sigma\ell \int_{-\infty}^{+\infty} (\hat{u}(\theta; c, \ell))^2 \,\mathrm{d}\theta = -\sigma\ell\mathscr{A}(c, \ell). \tag{3.7}$$

It follows immediately from this expression that

$$\det \begin{bmatrix} \mathscr{A}_c & \mathscr{A}_\ell \\ \mathscr{B}_c & \mathscr{B}_\ell \end{bmatrix} = -\sigma\mathscr{A}\mathscr{A}_c,$$

and therefore the condition (3.1) is satisfied precisely when $\sigma\mathscr{A}_c > 0$. When $\ell = 0$ $\mathscr{A}(c, \ell)|_{\ell=0}$ is the momentum of the solitary wave of the generalized KdV, and for this wave it is well-known that the sign of $\mathscr{A}_c(c, 0) = \mathrm{sign}(p-4)$. Therefore condition (3.1) leads immediately to the condition $\sigma(p-4) > 0$ for transverse instability of the solitary wave (3.4)–(3.5) when $\ell = 0$. This recovers the well-known result for transverse instability of the generalized KdV solitary wave (cf. Kadomtsev & Petviashvili 1970; Zakharov 1975; Ostrovsky & Shrira 1976; Alexander *et al.* 1997; Allen & Rowlands 1997). The case $\ell \neq 0$ can also be considered (i.e. apply condition (3.1) for finite $\ell$) but leads to the same result (see § 4).

For the case of longitudinal instability of KdV-type equations results are known even for $p = 4$. Pelinovsky & Grimshaw (1996) expand the Evans-type function for solitary waves to higher order and show that when $p = 4$ the KdV-type solitary wave is unstable. It is not difficult to show that this result carries over to the case of transverse instabilities.

## 4. Transverse instability problem for the KP equation

In this section we justify the application of the instability criterion (3.1) to the KP model in § 3, by showing that the KP model is a Hamiltonian system on a multi-symplectic structure and then deducing the form of the function $\mathscr{B}(c, \ell)$ from the transverse symplectic structure. This section also provides an example of how one goes about applying the theory in a particular example. In constructing the multi-symplectic structure of the KP equation, we follow Bridges (1999).

Starting with the form of the KP equation given in (3.2), introduce new variables

$$\boldsymbol{Z} = \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{pmatrix} \in \mathbb{R}^4 \times \mathbb{R}^4. \tag{4.1}$$

These coordinates will be defined as follows, and used to reformulate the generalized KP equation as a system of first-order PDEs. Define $q_1$, $q_2$, $q_3$ and $p_3$ by

$$u \stackrel{\text{def}}{=} p_3 = \frac{\partial}{\partial x} q_2 = \frac{\partial^2}{\partial x^2} q_1 \quad \text{and} \quad q_3 = \frac{\partial}{\partial x} p_3. \tag{4.2}$$

Define $p_1$, $p_2$ and $p_4$ by

$$\left.\begin{aligned} p_1 &= -\frac{\partial p_3}{\partial t} - \frac{\partial p_2}{\partial x} - \sigma \frac{\partial p_4}{\partial y} \\ p_2 &= f(p_3) + \frac{\partial q_2}{\partial t} + \frac{\partial q_3}{\partial x} - \frac{\partial q_4}{\partial y} \\ p_4 &= \frac{\partial q_2}{\partial y} + \frac{1}{\sigma}\frac{\partial q_4}{\partial x}. \end{aligned}\right\} \tag{4.3}$$

Then the KP equation (3.2) is recovered by

$$\frac{\partial p_4}{\partial x} - \frac{\partial p_3}{\partial y} = 0 \quad \text{and} \quad \frac{\partial p_1}{\partial x} = 0. \tag{4.4}$$

This can be verified by substituting (4.1)–(4.3) into the two equations in (4.4). Combining these equations leads to a system of eight first-order PDEs which can be written in the form

$$\mathbf{M} Z_t + \mathbf{K} Z_x + \mathbf{L} Z_y = \nabla S(\boldsymbol{Z}), \quad \boldsymbol{Z} \in \mathbb{R}^8, \tag{4.5}$$

where

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sigma \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$S(\boldsymbol{Z}) = p_1 q_2 + p_2 p_3 + \tfrac{1}{2}\sigma p_4^2 - \tfrac{1}{2} q_3^2 - F(p_3), \quad \text{where} \quad F(u) = \int_0^u f(s)\,\mathrm{d}s.$$

The three skew-symmetric matrices $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{L}$ are constant and therefore define closed two forms,

$$\omega^{(1)} = \mathbf{d}p_3 \wedge \mathbf{d}q_2, \quad \omega^{(2)} = \sum_{j=1}^{4} \mathbf{d}p_j \wedge \mathbf{d}q_j \quad \text{and} \quad \omega^{(3)} = \mathbf{d}q_4 \wedge \mathbf{d}p_3 + \sigma \mathbf{d}p_4 \wedge \mathbf{d}q_2.$$

Moreover, $\mathbf{J}(c,\ell) = \mathbf{K} - c\mathbf{M} + \ell\mathbf{L}$ is non-degenerate for any $(c,\ell) \in \mathbb{R}^2$.

The functions $\mathscr{A}$ and $\mathscr{B}$ needed for the instability theory are deduced as follows. According to the definition in (2.8),

$$\begin{aligned}
\mathscr{A}(c,\ell) &= -\int_{-\infty}^{+\infty} \tfrac{1}{2}\omega^{(1)}(\hat{Z}_\theta, \hat{\boldsymbol{Z}})\,\mathrm{d}\theta = -\int_{-\infty}^{+\infty} \tfrac{1}{2}\langle \mathbf{M}\hat{Z}_\theta, \hat{\boldsymbol{Z}}\rangle\,\mathrm{d}\theta \\
&= -\frac{1}{2}\int_{-\infty}^{+\infty} \left( \hat{q}_2 \frac{\mathrm{d}}{\mathrm{d}\theta}\hat{p}_3 - \hat{p}_3 \frac{\mathrm{d}}{\mathrm{d}\theta}\hat{q}_2 \right) \mathrm{d}\theta \\
&= -\int_{-\infty}^{+\infty} \hat{p}_3 \frac{\mathrm{d}}{\mathrm{d}\theta}\hat{q}_2\,\mathrm{d}\theta \quad \text{(after integration by parts).}
\end{aligned}$$

In the integration by parts, the hypothesis (2.4) has been used. Now, noting that $\hat{p}_3 = \hat{u}$ and $(\mathrm{d}/\mathrm{d}\theta)\hat{q}_2 = \hat{p}_3$,

$$\mathscr{A}(c,\ell) = -\int_{-\infty}^{+\infty} \hat{u}^2\,\mathrm{d}\theta,$$

which is the expression used in § 3 (see equation (3.6)). The minus sign is a consequence of the choice of coordinates and the choice of moving frame. (Once the sign is fixed, it carries through the stability analysis: the instability criterion (3.1) is independent of this choice.)

A similar argument leads to an expression for $\mathscr{B}$; using (2.9),

$$\begin{aligned}
\mathscr{B}(c,\ell) &= \int_{-\infty}^{+\infty} \tfrac{1}{2}\omega^{(3)}(\hat{Z}_\theta, \hat{\boldsymbol{Z}})\,\mathrm{d}\theta = \int_{-\infty}^{+\infty} \tfrac{1}{2}\langle \mathbf{L}\hat{Z}_\theta, \hat{\boldsymbol{Z}}\rangle\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \left( \sigma\left( \hat{p}_4 \frac{\mathrm{d}}{\mathrm{d}\theta}\hat{q}_2 - \hat{q}_2 \frac{\mathrm{d}}{\mathrm{d}\theta}\hat{p}_4 \right) + \hat{q}_4 \frac{\mathrm{d}}{\mathrm{d}\theta}\hat{p}_3 - \hat{p}_3 \frac{\mathrm{d}}{\mathrm{d}\theta}\hat{q}_4 \right) \mathrm{d}\theta.
\end{aligned}$$

For the solitary-wave state (3.4)–(3.5), it is straightforward to show that $\hat{q}_4 = 0$ and $\hat{p}_4 = \ell\hat{p}_3 = \ell\hat{u}$. Therefore, after an integration by parts (using the hypothesis (2.4)),

$$\mathscr{B}(c,\ell) = \sigma\ell \int_{-\infty}^{+\infty} \hat{u}^2\,\mathrm{d}\theta,$$

recovering the function used in the instability calculation in § 3. The condition (3.1) applies to any $f(u)$ in (3.2) for which a solitary-wave solution exists. We finish this section by giving the complete calculation of the determinant (3.1) for finite $\ell$, when $f(u) = u^{p+1}/(p+1)$.

With the expression for the solitary wave (3.4)–(3.5), the functions $\mathscr{A}$ and $\mathscr{B}$ are computed as follows

$$\mathscr{A}(c,\ell) = -\int_{-\infty}^{+\infty} \hat{u}^2\,\mathrm{d}\theta = -\frac{a^2}{\gamma} m(p),$$

where $m(p) = \int_{-\infty}^{+\infty} \operatorname{sech}^{4/p}\xi \, d\xi$ is independent of $c$ and $\ell$, and clearly $\mathscr{B}(c,\ell) = -\sigma\ell\mathscr{A}(c,\ell)$. Hence

$$\det\begin{bmatrix} \mathscr{A}_c & \mathscr{A}_\ell \\ \mathscr{B}_c & \mathscr{B}_\ell \end{bmatrix} = \det\begin{bmatrix} \mathscr{A}_c & \mathscr{A}_\ell \\ -\sigma\ell\mathscr{A}_c & -\sigma\mathscr{A} - \sigma\ell\mathscr{A}_\ell \end{bmatrix} = -\sigma\mathscr{A}\mathscr{A}_c = \sigma|\mathscr{A}|\mathscr{A}_c,$$

since $\mathscr{A} < 0$. Therefore any solitary wave with $2c - \sigma\ell^2 > 0$ is transverse unstable if

$$\sigma\mathscr{A}_c > 0. \tag{4.6}$$

For any admissible $(c,\ell)$ the second term in this expression is

$$\mathscr{A}_c = \frac{a^2 m(p)}{p\gamma b^2}(p-4),$$

which is essentially the same as the case $\ell = 0$. In other words the transverse instability for KP is independent of the angle between the direction of travel of the basic state and the $x$-axis. However, this is a special property of the KP equation and is not true in general. For example, the transverse instability of the basic solitary-wave state of the Zakharov–Kuznetsov equation shows a strong dependence on the angle between the direction of propagation of the wave and the $x$-axis (cf. Allen & Rowlands 1993).

## 5. Implications of a transverse reflection symmetry

It is evident that equations like the KP equation and the water-wave equations have a reflection symmetry in the transverse direction. The immediate implication of this reversibility for the KP equation is that $u(x,-y,t)$ is a solution whenever $u(x,y,t)$ is a solution. This reflection symmetry will also arise in some form in the functions $\mathscr{A}(c,\ell)$ and $\mathscr{B}(c,\ell)$. In fact, we will prove that an implication of transverse reflection is the following property:

$$\mathscr{A}(c,-\ell) = \mathscr{A}(c,\ell) \quad \text{and} \quad \mathscr{B}(c,-\ell) = -\mathscr{B}(c,\ell). \tag{5.1}$$

To prove this we need a precise definition of transverse reversibility. A system in multi-symplectic form,

$$\mathbf{M}Z_t + \mathbf{K}Z_x + \mathbf{L}Z_y = \nabla S(\boldsymbol{Z}), \quad \boldsymbol{Z} \in \mathbb{X}, \tag{5.2}$$

is *transverse reversible* if there exists a *reversor* $\mathbf{R}$ acting on $\mathbb{X}$ satisfying

$$\mathbf{RM} = \mathbf{MR}, \quad \mathbf{RK} = \mathbf{KR}, \quad \mathbf{RL} = -\mathbf{LR}, \quad \text{and} \quad S(\mathbf{R}\boldsymbol{Z}) = S(\boldsymbol{Z}). \tag{5.3}$$

An operator $\mathbf{R}$ is a *reversor* if it is an involution and an isometry,

$$\mathbf{RR} = \mathbf{I} \quad \text{and} \quad \langle \mathbf{R}\boldsymbol{U}, \mathbf{R}\boldsymbol{W}\rangle = \langle \boldsymbol{U}, \boldsymbol{W}\rangle, \quad \forall \boldsymbol{U}, \boldsymbol{W} \in \mathbb{X}. \tag{5.4}$$

Act on (5.2) with $\mathbf{R}$,

$$\mathbf{RM}Z_t + \mathbf{RK}Z_x + \mathbf{RL}Z_y = \mathbf{R}\nabla S(\boldsymbol{Z}),$$

and use (5.3),

$$\mathbf{M}(\mathbf{R}\boldsymbol{Z})_t + \mathbf{K}(\mathbf{R}\boldsymbol{Z})_x - \mathbf{L}(\mathbf{R}\boldsymbol{Z})_y = \nabla S(\mathbf{R}\boldsymbol{Z}).$$

An immediate implication is that $\mathbf{R}\boldsymbol{Z}(x,-y,t)$ is a solution of (5.2) whenever $\boldsymbol{Z}(x,y,t)$ is a solution.

Before proving (5.1), we will construct a reversor for the KP equation; let

$$\mathbf{R}_{\mathrm{KP}} = \operatorname{diag}[1,1,1,-1,1,1,1,-1]. \tag{5.5}$$

It is evident that $\mathbf{R}_{\rm KP}$ is an involution and an isometry – with respect to the standard inner product on $\mathbb{R}^8$ – and it is straightforward to verify that $\mathbf{R}_{\rm KP}$ satisfies the relations (5.3), using the definitions of $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{L}$ for the KP equation given in § 4. Therefore the KP equation has the transverse reversibility property. For the water-wave problem a transverse reversor will be constructed in § 6.

To verify (5.1), we start with the definitions

$$\mathscr{A}(c,\ell) = -\int_{-\infty}^{+\infty} \tfrac{1}{2}\langle \mathbf{M}\hat{Z}_\theta, \hat{\boldsymbol{Z}}\rangle\,\mathrm{d}\theta \quad \text{and} \quad \mathscr{B}(c,\ell) = \int_{-\infty}^{+\infty} \tfrac{1}{2}\langle \mathbf{L}\hat{Z}_\theta, \hat{\boldsymbol{Z}}\rangle\,\mathrm{d}\theta, \tag{5.6}$$

with $\hat{\boldsymbol{Z}}(\theta,c,l)$ satisfying $\mathbf{J}(c,\ell)\hat{Z}_\theta = \nabla S(\hat{\boldsymbol{Z}})$, or

$$\mathbf{K}\hat{Z}_\theta - c\mathbf{M}\hat{Z}_\theta + \ell\mathbf{L}\hat{Z}_\theta = \nabla S(\hat{\boldsymbol{Z}}). \tag{5.7}$$

Act on this equation with $\mathbf{R}$ and use (5.3),

$$\mathbf{K}(\mathbf{R}\hat{\boldsymbol{Z}})_\theta - c\mathbf{M}(\mathbf{R}\hat{\boldsymbol{Z}})_\theta - \ell\mathbf{L}(\mathbf{R}\hat{\boldsymbol{Z}})_\theta = \nabla S(\mathbf{R}\hat{\boldsymbol{Z}}).$$

For each $(c,\ell)$, suppose $\hat{\boldsymbol{Z}}(\theta,c,\ell)$ is a unique solution of (5.7). It then follows that

$$\mathbf{R}\hat{\boldsymbol{Z}}(\theta,c,\ell) = \hat{\boldsymbol{Z}}(\theta,c,-\ell). \tag{5.8}$$

Now, use this identity in (5.6),

$$\begin{aligned}
\mathscr{A}(c,-\ell) &= -\int_{-\infty}^{+\infty} \frac{1}{2}\left\langle \mathbf{M}\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{\boldsymbol{Z}}(\theta,c,-\ell), \hat{\boldsymbol{Z}}(\theta,c,-\ell)\right\rangle \mathrm{d}\theta \\
&= -\int_{-\infty}^{+\infty} \frac{1}{2}\left\langle \mathbf{M}\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbf{R}\hat{\boldsymbol{Z}}(\theta,c,\ell), \mathbf{R}\hat{\boldsymbol{Z}}(\theta,c,\ell)\right\rangle \mathrm{d}\theta \\
&= -\int_{-\infty}^{+\infty} \frac{1}{2}\left\langle \mathbf{R}\mathbf{M}\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{\boldsymbol{Z}}(\theta,c,\ell), \mathbf{R}\hat{\boldsymbol{Z}}(\theta,c,\ell)\right\rangle \mathrm{d}\theta \\
&= -\int_{-\infty}^{+\infty} \frac{1}{2}\left\langle \mathbf{M}\hat{Z}_\theta, \hat{\boldsymbol{Z}}\right\rangle \mathrm{d}\theta = \mathscr{A}(c,\ell).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathscr{B}(c,-\ell) &= \int_{-\infty}^{+\infty} \frac{1}{2}\left\langle \mathbf{L}\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{\boldsymbol{Z}}(\theta,c,-\ell), \hat{\boldsymbol{Z}}(\theta,c,-\ell)\right\rangle \mathrm{d}\theta \\
&= \int_{-\infty}^{+\infty} \frac{1}{2}\left\langle \mathbf{L}\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbf{R}\hat{\boldsymbol{Z}}(\theta,c,\ell), \mathbf{R}\hat{\boldsymbol{Z}}(\theta,c,\ell)\right\rangle \mathrm{d}\theta \\
&= -\int_{-\infty}^{+\infty} \frac{1}{2}\left\langle \mathbf{R}\mathbf{L}\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{\boldsymbol{Z}}(\theta,c,\ell), \mathbf{R}\hat{\boldsymbol{Z}}(\theta,c,\ell)\right\rangle \mathrm{d}\theta \\
&= -\int_{-\infty}^{+\infty} \frac{1}{2}\left\langle \mathbf{L}\hat{Z}_\theta, \hat{\boldsymbol{Z}}\right\rangle \mathrm{d}\theta = -\mathscr{B}(c,\ell).
\end{aligned}$$

This completes the proof of (5.1). An immediate and important consequence of this result is that

$$\lim_{\ell\to 0}\det\begin{bmatrix} \mathscr{A}_c & \mathscr{A}_\ell \\ \mathscr{B}_c & \mathscr{B}_\ell \end{bmatrix} = \mathscr{A}_c(c,0)\hat{B}(c) \quad \text{where} \quad \hat{B}(c) = \lim_{\ell\to 0}\frac{1}{\ell}\mathscr{B}(c,\ell). \tag{5.9}$$

The functions $\mathscr{A}(c, 0)$ and $\hat{B}(c)$ can be determined from the $y$-independent state. In other words, when the system has a reflection symmetry in the transverse direction, the transverse instability determinant can be determined from properties of the $\ell$-independent state. For example, for the KP equation, $\hat{\mathscr{B}}(c) = \sigma|\mathscr{A}|$, and so the result (4.6) follows immediately from (5.9).

## 6. Solitary-wave states of the water-wave problem

The purpose of this section is threefold: (*a*) to show that the instability condition of § 3 formally applies to the full water-wave equations, (*b*) to introduce a new functional relevant to water waves, particularly the transverse instability of water waves, based on the transverse symplectic structure, and (*c*) to show that the analogue of the functional $\hat{\mathscr{B}}(c)$ for water waves is strictly positive. Point (*c*) is the most intriguing, because qualitative stability results can be deduced without calculation.

Verification of the above statements reduces essentially to showing that the water-wave problem can be formulated as a Hamiltonian system on a multi-symplectic structure, and this has been shown in Bridges (1996, 1997*a*). However, in that multi-symplectic formulation for water waves, the skew-symmetric operators $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{L}$ depend explicitly on the dependent variables. The theory of this paper can be modified to treat this case. However, it turns out to be easier to transform the water-wave problem so that the three operators $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{L}$ reduce to constant operators, and then the theory of this paper applies in a straightforward way.

Consider the water-wave problem in standard form for irrotational inviscid flow with a single-valued free surface. The principal dependent variables are the velocity potential $\phi(x, y, z, t)$ and free-surface position $\eta(x, y, t)$. The function $\phi$ is required to be harmonic in the interior and to satisfy the kinematic and dynamic boundary conditions at the free surface. These equations can be formulated as a Hamiltonian system on a multi-symplectic structure,

$$\mathbf{M}Z_t + \mathbf{K}(\mathscr{u})Z_x + \mathbf{L}(\mathscr{v})Z_y = \nabla S(\boldsymbol{Z}), \tag{6.1}$$

by taking

$$\boldsymbol{Z} = \left\{ \begin{array}{c} \Phi \\ \eta \\ w_1 \\ w_2 \\ \phi \\ u \\ v \end{array} \right\} \text{where} \left\{ \begin{array}{l} \Phi = \phi|_{z=\eta}, \\ u = \phi_x, \\ v = \phi_y, \\ \mathscr{u} = u|_{z=\eta}, \\ \mathscr{v} = v|_{z=\eta}. \end{array} \right. \tag{6.2}$$

and

$$w_1 = \frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_y^2}} \quad \text{and} \quad w_2 = \frac{\eta_y}{\sqrt{1 + \eta_x^2 + \eta_y^2}}.$$

Choose $z = 0$ at the fluid bottom for simplicity and then $0 \leqslant z \leqslant \eta(x, y, t)$. The functional $S(\boldsymbol{Z})$ in the above system is

$$S(\boldsymbol{Z}) = \frac{1}{2}\int_0^\eta (u^2 + v^2 - \phi_z^2)\,\mathrm{d}z - \tfrac{1}{2}g\eta^2 + \tau(1 - \sqrt{1 - w_1^2 - w_2^2}) \tag{6.3}$$

where $\tau$ is the coefficient of surface tension.

Although the symplectic operators are not constant, they are exact and therefore closed, and the one-forms associated with the three symplectic forms are

$$\alpha^{(1)} = -\Phi \mathbf{d}\eta, \quad \alpha^{(2)} = \int_0^\eta u \mathbf{d}\phi \, \mathrm{d}z + \tau w_1 \mathbf{d}\eta, \quad \alpha^{(3)} = \int_0^\eta v \mathbf{d}\phi \, \mathrm{d}z + \tau w_2 \mathbf{d}\eta,$$

and from these we find that $\omega^{(j)} = \mathbf{d}\alpha^{(j)}$ for $j = 1, 2, 3$ or

$$\omega^{(1)} = \mathbf{d}\alpha^{(1)} = -\mathbf{d}\Phi \wedge \mathbf{d}\eta,$$

$$\omega^{(2)} = \mathbf{d}\alpha^{(2)} = \int_0^\eta \mathbf{d}u \wedge \mathbf{d}\phi \, \mathrm{d}z + \mathscr{u}\mathbf{d}\eta \wedge \mathbf{d}\Phi + \tau \mathbf{d}w_1 \wedge \mathbf{d}\eta,$$

$$\omega^{(3)} = \mathbf{d}\alpha^{(3)} = \int_0^\eta \mathbf{d}v \wedge \mathbf{d}\phi \, \mathrm{d}z + \mathscr{v}\mathbf{d}\eta \wedge \mathbf{d}\Phi + \tau \mathbf{d}w_2 \wedge \mathbf{d}\eta.$$

The relation with the skew-symmetric operators $\mathbf{M}$, $\mathbf{K}(\mathscr{u})$ and $\mathbf{L}(\mathscr{v})$ is as follows. Introduce the inner product

$$\langle \boldsymbol{U}, \boldsymbol{V} \rangle_\eta = U_1 V_1 + U_2 V_2 + U_3 V_3 + U_4 V_4 + \int_0^\eta [U_5 V_5 + U_6 V_6 + U_7 V_7] \, \mathrm{d}z \tag{6.4}$$

then it is easily verified that

$$\omega^{(1)}(\boldsymbol{U}, \boldsymbol{V}) = \langle \mathbf{M}\boldsymbol{U}, \boldsymbol{V} \rangle_\eta, \quad \omega^{(2)}(\boldsymbol{U}, \boldsymbol{V}) = \langle \mathbf{K}(\mathscr{u})\boldsymbol{U}, \boldsymbol{V} \rangle_\eta, \quad \omega^{(3)}(\boldsymbol{U}, \boldsymbol{V}) = \langle \mathbf{L}(\mathscr{v})\boldsymbol{U}, \boldsymbol{V} \rangle_\eta.$$

Explicit expressions for the operators $\mathbf{M}$, $\mathbf{K}(\mathscr{u})$ and $\mathbf{L}(\mathscr{v})$ are given in Bridges (1996, § 2).

The water-wave equations have a transverse reflection symmetry. In the multi-symplectic coordinates (6.2), the reversor is

$$\mathbf{R}_{\mathrm{WW}} = \mathrm{diag}[1, 1, 1, -1, 1, 1, -1].$$

This operator is clearly an involution and an isometry with respect to the inner product (6.4). Moreover, given the explicit expressions for $\mathbf{M}$, $\mathbf{K}(\mathscr{u})$ and $\mathbf{L}(\mathscr{v})$ it is straightforward to verify the conditions (5.3). The theory in § 7 requires that $\mathbf{M}$, $\mathbf{K}(\mathscr{u})$ and $\mathbf{L}(\mathscr{v})$ are constant operators. The above multi-symplectic structure can be transformed so that these operators are constant, and the details of the transformation are given in the Appendix. In the remainder of this section, the implications of the multi-symplectic structure of the water-wave problem for transverse instability are considered.

Given any solitary-wave solution, denoted by $\hat{\boldsymbol{Z}}(\theta, z; c, \ell)$, of the water-wave problem, which is differentiable, satisfies (6.1) and decays exponentially as $\theta \to \pm\infty$, it can be characterized formally as a critical point of a constrained variational principle as in § 2, where $\theta = x - ct + \ell y + \theta_0$. The governing equation for this solitary wave is a generalization of (2.5):

$$\mathbf{J}(c, \ell)\hat{Z}_\theta = \nabla S(\hat{\boldsymbol{Z}}), \quad \hat{\boldsymbol{Z}} \in \mathbb{X}, \tag{6.5}$$

where $\mathbf{J}(c, \ell) = \mathbf{K}(\mathscr{u}) - c\mathbf{M} + \ell \mathbf{L}(\mathscr{v})$, and in this case the solitary wave when it exists is a homoclinic orbit of this symplectic PDE on an infinite-dimensional space, $\mathbb{X}$.

The form of the functions $\mathscr{A}$ and $\mathscr{B}$ evaluated on a solitary wave follow immediately from the multi-symplectic structure,

$$\mathscr{A}(\hat{\boldsymbol{Z}}) = -\int_{-\infty}^{+\infty} \tfrac{1}{2} \langle \mathbf{M}\hat{Z}_\theta, \hat{\boldsymbol{Z}} \rangle_\eta \, \mathrm{d}\theta = -\int_{-\infty}^{+\infty} \tfrac{1}{2} \omega^{(1)}(\hat{Z}_\theta, \hat{\boldsymbol{Z}}) \, \mathrm{d}\theta, \tag{6.6}$$

and

$$\mathscr{B}(\hat{\boldsymbol{Z}}) = \int_{-\infty}^{+\infty} \tfrac{1}{2}\langle \mathbf{L}(\upsilon)\hat{Z}_\theta, \hat{\boldsymbol{Z}}\rangle \,\mathrm{d}\theta = \int_{-\infty}^{+\infty} \tfrac{1}{2}\omega^{(3)}(\hat{Z}_\theta, \hat{\boldsymbol{Z}})\,\mathrm{d}\theta. \tag{6.7}$$

Evaluating these expressions, and using (6.2), results in

$$\mathscr{A}(\hat{\boldsymbol{Z}}) = \int_{-\infty}^{+\infty} \hat{\Phi}\hat{\eta}_\theta \,\mathrm{d}\theta, \quad \text{and} \quad \mathscr{B}(\hat{\boldsymbol{Z}}) = \int_{-\infty}^{+\infty} \left( \tau \hat{w}_2 \hat{\eta}_\theta + \int_0^{\hat{\eta}} \hat{v}\hat{\phi}_\theta \,\mathrm{d}z \right) \mathrm{d}\theta. \tag{6.8}$$

The first expression is (minus) the familiar impulse or momentum for water waves evaluated on a solitary wave, but here also considered as a function of $\ell$, as well as $c$. The dependence of the momentum or impulse for solitary gravity waves as a function of $c$ (but not as a function of $\ell$) has been studied in the water-wave literature (cf. Longuet-Higgins 1974) including its consequences for longitudinal instability of solitary waves (cf. Tanaka 1986).

The second functional is new. For the solitary wave,

$$\hat{v} = \ell\hat{\phi}_\theta \quad \text{and} \quad \hat{w}_2 = \frac{\ell\hat{\eta}_\theta}{\sqrt{1+(1+\ell^2)\hat{\eta}_\theta^2}},$$

and therefore

$$\mathscr{B}(\hat{\boldsymbol{Z}}) = \ell \int_{-\infty}^{+\infty} \left( \tau \frac{\hat{\eta}_\theta^2}{\sqrt{1+(1+\ell^2)\hat{\eta}_\theta^2}} + \int_0^{\hat{\eta}} (\hat{\phi}_\theta)^2 \,\mathrm{d}z \right) \mathrm{d}\theta. \tag{6.9}$$

There is no obvious physical significance of this functional. For example, when surface tension vanishes, it reduces to

$$\mathscr{B}(\hat{\boldsymbol{Z}}) = \ell \int_{-\infty}^{+\infty} \int_0^{\hat{\eta}} (\hat{\phi}_\theta)^2 \mathrm{d}z \,\mathrm{d}\theta.$$

It is not related to the transverse momentum. It is similar to a kinetic energy, but is lacking the vertical velocity squared in order to be related to twice the true kinetic energy. It is an example of a functional whose importance has been dictated by the mathematical structure of the equations rather than physical considerations.

It can be immediately deduced from the expression for $\mathscr{B}(\hat{Z})$ in equation (6.9) – regardless of whether the solitary wave is a gravity wave or capillary–gravity wave – that

$$\hat{\mathscr{B}}(c) = \lim_{\ell\to 0} \frac{1}{\ell}\mathscr{B}(c,\ell) > 0.$$

This result is remarkable, because – when combined with the fact that the water-wave problem has a transverse reflection symmetry – it says that transverse instability for the case $\ell = 0$ is determined precisely by the sign of $\mathscr{A}_c$. But

$$\mathrm{sign}\mathscr{A}_c(c,0) = -\mathrm{sign} I_c(c)$$

where $I_c(c)$ is the momentum for solitary wave travelling in the $x$-direction. Therefore a sufficient condition for transverse instability of solitary water waves is

$$\frac{\mathrm{d}I}{\mathrm{d}c} < 0.$$

For classical gravity solitary waves with monotone decay of the tails as $\theta \to \pm\infty$, it is known from the numerical results of Longuet-Higgins (1974) that $\mathrm{d}I/\mathrm{d}c > 0$ for finite-amplitude solitary waves up to almost the highest wave, as shown schematically in figure 1. The present result shows that this solitary wave is also unstable when $\mathrm{d}I/\mathrm{d}c < 0$ to weakly transverse perturbations as well.

From the work of Tanaka *et al.* (1987), it is known that the longitudinal instabilities that arise at the point where $\mathrm{d}I/\mathrm{d}c$ first changes sign can lead to wave breaking. Now, the above theory shows that transverse instability also occurs at the same point. Therefore, if the initial data in a numerical calculation such as Tanaka *et al.* (1987) were chosen to be an unstable eigenfunction with non-zero transverse component, it is reasonable to conjecture that it would lead to wave breaking with transverse variation. This would be a natural mechanism for the appearance of three-dimensionality in wave breaking.

The condition $\mathrm{d}I/\mathrm{d}c < 0$ implying transverse instabilty also applies to capillary–gravity waves; indeed, given any solitary wave of the water-wave problem of the form $\hat{\boldsymbol{Z}}(\theta, z; c, \ell)$ as in (6.5) with $\ell = 0$, a negative sign for $\mathrm{d}I/\mathrm{d}c$ indicates immediately that it is transverse unstable. However, there is little information in the literature about the value of $I(c)$ along branches of capillary–gravity solitary waves.

One of the main conclusions one can draw from the above theory is the importance of plotting the surfaces $\mathscr{A}(c, \ell)$ and $\mathscr{B}(c, \ell)$ as functions of $c$ and $\ell$. These plots would be a natural generalization of plots of the momentum or impulse against the wave speed $c$ as in figure 1.

Given the graphs of $\mathscr{A}(c, \ell)$ and $\mathscr{B}(c, \ell)$, the idea would be to look for points where $\nabla\mathscr{A}$ and the gradient of $\nabla\mathscr{B}$ become collinear (where $\nabla$ represents the gradient with respect to $c$ and $\ell$). These points are neutral points and candidates for a change from transverse stability to transverse instability.

## 7. Formulating the stability problem for transverse perturbations

The linear stability problem for transverse perturbations is formulated as follows. Let

$$\boldsymbol{Z}(x, y, t) = \hat{\boldsymbol{Z}}(\theta; c, \ell) + \hat{\boldsymbol{U}}(\theta, y, t), \tag{7.1}$$

where $\hat{\boldsymbol{Z}}(\theta; c, \ell)$ is a solitary wave solution of the form (2.3)–(2.4), and any cross-section variables (such as $z$ in the water-wave problem) are suppressed. Substitute (7.1) into (2.1) and formally linearize about $\hat{\boldsymbol{Z}}$,

$$\mathbf{M}\hat{U}_t + \mathbf{J}(c, \ell)\hat{U}_\theta + \mathbf{L}\hat{U}_y = D^2S(\hat{\boldsymbol{Z}})\hat{\boldsymbol{U}}, \tag{7.2}$$

where $D^2S(\hat{\boldsymbol{Z}})$ is the Hessian of $S$ evaluated at $\hat{\boldsymbol{Z}}$.

Since the coefficients in this PDE are independent of $y$ and $t$, consider solutions of (7.2) of the form

$$\hat{\boldsymbol{U}}(\theta, y, t) = \mathrm{Re}[\boldsymbol{U}(\theta; \lambda, \beta)\mathrm{e}^{\mathrm{i}\beta y}\mathrm{e}^{\lambda t}]; \tag{7.3}$$

then the complex vector-valued function $\boldsymbol{U}(\theta; \lambda; \beta)$ satisfies the differential equation

$$\mathscr{L}\boldsymbol{U} = \lambda\mathbf{M}\boldsymbol{U} + \mathrm{i}\beta\mathbf{L}\boldsymbol{U}, \quad \boldsymbol{U} \in \mathbb{Y}, \tag{7.4}$$

where $\mathbb{Y}$ is the complexification of $\mathbb{X}$, and

$$\mathscr{L} \stackrel{\mathrm{def}}{=} D^2S(\hat{\boldsymbol{Z}})\boldsymbol{U} - \mathbf{J}(c, \ell)U_\theta. \tag{7.5}$$

The basic state $\hat{\boldsymbol{Z}}(\theta; c, \ell)$ is said to be *linearly unstable* or *spectrally unstable* if, for some $\beta \in \mathbb{R}$, there exists $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$ such that (7.4) has a solution $\boldsymbol{U}(\theta; \lambda, \beta)$ which is bounded for all $\theta \in \mathbb{R}$ and decays exponentially to zero as

$\theta \to \pm\infty$. If $\ell = \beta = 0$ then we recover the case of one-dimensional instabilities of a one-dimensional solitary wave travelling in the $x$-direction.

The idea is to derive an instability criterion for $|\lambda| + |\beta|$ sufficiently small by projecting the right-hand side of (7.4) onto the kernel of $\mathscr{L}$.

By differentiating (2.5), it is clear that $\hat{Z}_\theta \in \mathrm{Ker}(\mathscr{L})$. We take as a hypothesis that the kernel is not larger; in other words there are no other solutions which are square integrable and in the kernel of $\mathscr{L}$. The kernel may be larger if symmetries are present, or for particular values of the parameters, but generically we can expect this condition to be satisfied.

Formally the operator $\mathscr{L}$ is self-adjoint, and therefore the condition for (7.4) to be solvable is that the right-hand side be in the range of the operator $\mathscr{L}$. This condition leads to

$$\Delta(\lambda, \beta) = [\![\hat{Z}_\theta, \lambda \mathbf{M}\boldsymbol{U} + \mathrm{i}\beta \mathbf{L}\boldsymbol{U}]\!] = 0, \tag{7.6}$$

with $U$ satisfying (7.4). The inner product is defined by

$$[\![\boldsymbol{U}, \boldsymbol{V}]\!] \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} \langle \boldsymbol{U}(\theta), \boldsymbol{V}(\theta)\rangle \mathrm{d}\theta, \tag{7.7}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{Y}$.

The central result needed for the theory is that the complex function $\Delta(\lambda, \beta)$ has the following Taylor expansion for $|\lambda| + |\beta|$ sufficiently small:

$$\Delta(\lambda, \beta) = \mathbb{C}(-\mathscr{A}_c\lambda^2 + (\mathscr{A}_\ell + \mathscr{B}_c)\mathrm{i}\lambda\beta + \mathscr{B}_\ell\beta^2) + o(|\lambda|^2 + |\beta|^2) \quad \text{as} \quad |\lambda| + |\beta| \to 0, \tag{7.8}$$

where $\mathbb{C}$ is a non-zero complex constant.

The instability criterion (3.1) can be deduced from this expression immediately. Solving $\Delta(\lambda, \beta) = 0$ for $|\lambda| + |\beta|$ sufficiently small leads to

$$\lambda = \mathrm{i}\frac{\mathscr{A}_\ell}{\mathscr{A}_c}\beta \pm \frac{\beta}{\mathscr{A}_c}\sqrt{\mathscr{A}_c\mathscr{B}_\ell - \mathscr{A}_\ell\mathscr{B}_c} + O(\beta^2) \tag{7.9}$$

(noting that $\mathscr{A}_\ell = \mathscr{B}_c$) and therefore if $\beta \neq 0$ but $|\beta| \ll 1$ and the determinant (3.1) is positive there is an unstable exponent $\lambda$ with positive real part, with the magnitude of the growth rate of order $|\beta|$. Reference to equation (7.3) shows that the unstable eigenfunction is travelling in a direction transverse to the $x$-axis.

The Taylor expansion (7.8) is verified as follows. Differentiation of (2.5) with respect to $x$, $c$ and $\ell$ leads to the following identities:

$$\begin{aligned}
\mathbf{J}(c, \ell)\frac{\partial}{\partial\theta}\hat{Z}_\theta &= D^2S(\hat{\boldsymbol{Z}})\hat{Z}_\theta, \\
\mathbf{J}(c, \ell)\frac{\partial}{\partial\theta}\hat{Z}_c &= D^2S(\hat{\boldsymbol{Z}})\hat{Z}_c + \mathbf{M}\hat{Z}_\theta, \\
\mathbf{J}(c, \ell)\frac{\partial}{\partial\theta}\hat{Z}_\ell &= D^2S(\hat{\boldsymbol{Z}})\hat{Z}_\ell - \mathbf{L}\hat{Z}_\theta.
\end{aligned}$$

Using the definition of $\mathscr{L}$ in (7.5) these three equations can be written

$$\mathscr{L}\hat{Z}_\theta = 0, \quad \mathscr{L}\hat{Z}_c = -\mathbf{M}\hat{Z}_\theta \quad \text{and} \quad \mathscr{L}\hat{Z}_\ell = \mathbf{L}\hat{Z}_\theta. \tag{7.10}$$

Therefore, by letting

$$\boldsymbol{U}(\theta; \lambda, \beta) = \mathbb{C}(\hat{Z}_\theta - \lambda\hat{Z}_c + \mathrm{i}\beta\hat{Z}_\ell + O(|\lambda|^2 + |\beta|^2)), \tag{7.11}$$

it is clear that $\boldsymbol{U}(\theta; \lambda, \beta)$ satisfies (7.4) to order $|\lambda|^2 + |\beta|^2$. Substitution of (7.11) into

(7.6) then leads to

$$\Delta(\lambda,\beta) = \mathbb{C}[\![\hat{Z}_\theta, \lambda\mathbf{M}(\hat{Z}_\theta - \lambda\hat{Z}_c + \mathrm{i}\beta\hat{Z}_\ell)]\!] + \mathbb{C}[\![\hat{Z}_\theta, \mathrm{i}\beta\mathbf{L}(\hat{Z}_\theta - \lambda\hat{Z}_c + \mathrm{i}\beta\hat{Z}_\ell)]\!] + \cdots.$$

But, due to the skew-symmetry of $\mathbf{M}$ and $\mathbf{L}$,

$$[\![\hat{Z}_\theta, \mathbf{M}\hat{Z}_\theta]\!] = [\![\hat{Z}_\theta, \mathbf{L}\hat{Z}_\theta]\!] = 0.$$

Hence,

$$\Delta(\lambda,\beta) = \mathbb{C}\{-\lambda^2[\![\hat{Z}_\theta, \mathbf{M}\hat{Z}_c]\!] + \mathrm{i}\beta\lambda[\![\hat{Z}_\theta, \mathbf{M}\hat{Z}_\ell]\!] - \mathrm{i}\lambda\beta[\![\hat{Z}_\theta, \mathbf{L}\hat{Z}_c]\!] - \beta^2[\![\hat{Z}_\theta, \mathbf{L}\hat{Z}_\ell]\!]\} + \cdots. \quad (7.12)$$

It remains to evaluate the inner products in this expression. But

$$[\![\hat{Z}_\theta, \mathbf{M}\hat{Z}_c]\!] = \int_{-\infty}^{+\infty} \langle \hat{Z}_\theta, \mathbf{M}\hat{Z}_c \rangle \mathrm{d}\theta = -\int_{-\infty}^{+\infty} \omega^{(1)}(\hat{Z}_\theta, \hat{Z}_c)\,\mathrm{d}\theta = \mathscr{A}_c,$$

using (2.14). Similar evaluation of the other three inner products leads to

$$\begin{pmatrix} +[\![\hat{Z}_\theta, \mathbf{M}\hat{Z}_c]\!] & +[\![\hat{Z}_\theta, \mathbf{M}\hat{Z}_\ell]\!] \\ -[\![\hat{Z}_\theta, \mathbf{L}\hat{Z}_c]\!] & -[\![\hat{Z}_\theta, \mathbf{L}\hat{Z}_\ell]\!] \end{pmatrix} = \begin{pmatrix} \mathscr{A}_c & \mathscr{A}_\ell \\ \mathscr{B}_c & \mathscr{B}_\ell \end{pmatrix},$$

using (2.14). Substitution of this expression into (7.12) then completes the verification of (7.8).

## 8. Concluding remarks

The main advantage of the instability condition (3.1) for transverse instability is that it is universal in the sense that it does not require an explicit solution or an explicit PDE: it relies only on geometric properties of a class of PDEs. It is a sufficient condition for instability. However, when the determinant has the opposite sign, it is a necessary but not sufficient condition for transverse stability. For example, for gravity solitary waves of the water-wave problem when $\mathrm{d}I/\mathrm{d}c > 0$ the condition suggests transverse stability but this is true only for transverse instabilities with long wavelength. It is possible in this case to have transverse instability with finite or short wavelength.

An asymptotic approach to extending the validity of the instability condition (3.1) to larger values of $\beta$ is to expand the characteristic function $\Delta(\lambda,\beta)$ in (7.8) to higher order in $\lambda$ and $\beta$. For example, this has been done for the case of longitudinal instabilities (i.e. $\beta = 0$) by Pelinovsky & Grimshaw (1996) and Skryabin (2000). These asymptotic expansions should also extend to the case $\beta \neq 0$. However, the higher-order terms in these expansions do not appear to be characterizable in terms of properties of the basic state, and would therefore require direct analysis of the linear-stability equation.

Many of the model equations proposed for water waves can be formulated as multi-symplectic systems and therefore the theory of this paper can be applied to solitary wave states. Examples are given below.

The $(2+1)$-dimensional Boussinesq equation,

$$u_{tt} - u_{xx} + 3(u^2)_{xx} + \varepsilon u_{xxxx} - u_{yy} = 0, \quad \varepsilon = \pm 1, \quad (8.1)$$

has been derived by Johnson (1996) as a model for interacting shallow-water waves, including solitary waves. It is in fact multi-symplectic, and therefore the theory here would be immediately applicable to $(c,\ell)$-parameter families of such waves (note however that the equation is ill-posed when $\varepsilon = -1$).

The nonlinear Schrödinger equation in two (or more) space dimensions,

$$-\mathrm{i}A_t + A_{xx} + \sigma A_{yy} + \varepsilon|A|^2 A = 0, \quad \varepsilon = \pm 1, \tag{8.2}$$

and its various generalizations such as fourth-order models and the Davey–Stewartson equation are important equations as models for water waves. In fact these equations have been widely studied for transverse instability phenomena (see the review by Kivshar & Pelinovsky 2000). These models are Hamiltonian generally and also multi-symplectic. However the theory presented here would require some interesting modification, since even the $y$-independent basic state can depend on two parameters, so it is expected that the transverse instability condition would be more complex than (3.1).

Coupled KP equations arise as models for the interaction between solitary waves, such as gap solitary waves in meterological flows (cf. Gottwald, Grimshaw & Malomed 1998). These systems have intriguing classes of solitary waves. The transverse instability has not been studied, but these systems have a multi-symplectic structure.

The Benney–Luke equation

$$u_{tt} - \Delta u + \mu(a\Delta^2 u - b\Delta u_{tt}) + \varepsilon(u_t\Delta u + \partial_t|\nabla u|^2) = 0, \tag{8.3}$$

where $\Delta = \partial_{xx} + \partial_{yy}$ and $b > 0$ and $a > 0$ are real parameters, first derived by Benney & Luke (1964) is a model for shallow-water waves more general than the Boussinesq models and the KP equation. In addition to solitary-wave solutions, it is known to have two-dimensional lump solitary waves as well (cf. Pego & Quinterro 1999). However, the stability problem has never been studied, but it is not difficult to show that this equation has a multi-symplectic structure.

The range of classes of solitary waves of the water-wave problem gets much richer when capillary forces are added, and very little is known about their stability, even the longitudinal stability. When the Bond number is near 1/3 and the Froude number is near unity, the relevant model equation is the fifth-order KdV equation. Therefore a good starting point for addressing the transverse instability of solitary waves near this critical point is the generalization of the fifth-order KdV to include transverse variation due to Hărăguş-Courcelle & Ilichev (1998),

$$(u_t + uu_x + u_{xxx} + \gamma u_{xxxxx})_x + \sigma u_{yy} = 0. \tag{8.4}$$

It is straightforward to show that this system is multi-symplectic (the multi-symplectic structure of this equation when $\sigma = 0$ is given in Bridges & Derks 2000). While it is known from the analysis of the KP equation that weakly nonlinear solitary waves are transverse unstable for Bond number greater than 1/3 (cf. Ablowitz & Segur 1979), this result does not hold for Bond number very near 1/3, where the equation (8.4) is relevant.

## Appendix. Transforming the free surface and multi-symplectic structure

In this appendix, the details of the transformation of the water-wave equations which leads to a constant multi-symplectic structure are given. The starting point is the multi-symplectic form of the water-wave equations given in § 6, and the goal is to transform them to the form (2.1), where the operators **M**, **K** and **L** are constant.

Flatten the free surface by introducing the new coordinate $\xi$ defined by

$$\xi = \frac{z}{\eta} \quad \text{with} \quad 0 \leqslant \xi \leqslant 1 \tag{A 1}$$

and then introduce new field variables which depend on $\xi$ instead of $z$,

$$\tilde{\boldsymbol{Z}} = \mathscr{F}(\boldsymbol{Z}) = (\Psi, \eta, W_1, W_2, \psi, U, V), \tag{A 2}$$

with the components of $\tilde{\boldsymbol{Z}}$ defined as follows:

$$\psi(x, y, \xi, t) = \phi(x, t, \xi\eta, t), \quad \Psi = \psi|_{\xi=1},$$

$$U(x, y, \xi, t) = \eta u(x, t, \xi\eta, t), \quad V(x, y, \xi, t) = \eta v(x, t, \xi\eta, t),$$

and

$$W_1 = \tau w_1 - \frac{1}{\eta}\int_0^\eta z u \phi_z \mathrm{d}z = \tau w_1 - \frac{1}{\eta}\int_0^1 \xi U \psi_\xi \mathrm{d}\xi,$$

$$W_2 = \tau w_2 - \frac{1}{\eta}\int_0^\eta z v \phi_z \mathrm{d}z = \tau w_2 - \frac{1}{\eta}\int_0^1 \xi V \psi_\xi \mathrm{d}\xi,$$

since

$$\psi_\xi = \eta\phi_z, \quad U_\xi = \eta^2 u_z \quad \text{and} \quad V_\xi = \eta^2 v_z.$$

The advantage of introducing this transformation in the multi-symplectic setting – instead of transforming the water-wave equations explicitly – is that the equations are automatically transformed by transforming the two-forms and the function $S$, which are much simpler to transform.

Under the transformation $\mathscr{F}$ the Hamiltonian function $S(\boldsymbol{Z})$ becomes

$$\begin{aligned}\tilde{S}(\tilde{\boldsymbol{Z}}) &= S(\mathscr{F}^{-1}(\tilde{\boldsymbol{Z}}))\\ &= \frac{1}{2\eta}\int_0^1 (U^2 + V^2 - \psi_\xi^2)\mathrm{d}\xi - \tfrac{1}{2}g\eta^2 + \tau\left(1 - \sqrt{1 - w_1^2 - w_2^2}\right),\end{aligned} \tag{A 3}$$

where it is to be understood that

$$w_1 = \frac{1}{\tau}\left(W_1 + \frac{1}{\eta}\int_0^1 \xi U \psi_\xi \mathrm{d}\xi\right) \quad \text{and} \quad w_2 = \frac{1}{\tau}\left(W_2 + \frac{1}{\eta}\int_0^1 \xi V \psi_\xi \mathrm{d}\xi\right). \tag{A 4}$$

It remains to transform the three differential forms, which will in turn automatically transform the operators $\mathbf{M}$, $\mathbf{K}(\mathscr{u})$ and $\mathbf{L}(\mathscr{v})$. The transformation of $\omega^{(1)}$ is straightforward since $\mathbf{d}\Phi = \mathbf{d}\Psi$ and so

$$\mathscr{F}^*\omega^{(1)} = -\mathbf{d}\Psi \wedge \mathbf{d}\eta.$$

In order to transform $\omega^{(2)}$ and $\omega^{(3)}$ we need the following identities:

$$\mathbf{d}\phi = \mathbf{d}\psi - \frac{\xi}{\eta}\psi_\xi \mathbf{d}\eta, \quad \eta\mathbf{d}u = \mathbf{d}U - \frac{1}{\eta}(U + \xi U_\xi)\mathbf{d}\eta, \quad \eta\mathbf{d}v = \mathbf{d}V - \frac{1}{\eta}(V + \xi V_\xi)\mathbf{d}\eta,$$

and

$$\tau\mathbf{d}w_1 = \mathbf{d}W_1 + \frac{1}{\eta}\int_0^1 \xi\psi_\xi \mathbf{d}U\,\mathrm{d}\xi + \frac{1}{\eta}U|_{\xi=1}\mathbf{d}\Psi - \frac{1}{\eta}\int_0^1 (U + \xi U_\xi)\mathbf{d}\psi\,\mathrm{d}\xi + (\cdots)\mathbf{d}\eta,$$

$$\tau\mathbf{d}w_2 = \mathbf{d}W_2 + \frac{1}{\eta}\int_0^1 \xi\psi_\xi \mathbf{d}V\,\mathrm{d}\xi + \frac{1}{\eta}V|_{\xi=1}\mathbf{d}\Psi - \frac{1}{\eta}\int_0^1 (V + \xi V_\xi)\mathbf{d}\psi\,\mathrm{d}\xi + (\cdots)\mathbf{d}\eta.$$

The terms proportional to $\mathbf{d}\eta$ in $\mathbf{d}w_1$ and $\mathbf{d}w_2$ are not recorded since they drop out of the wedge product $\mathbf{d}w_j \wedge \mathbf{d}\eta$ $(j = 1, 2)$. We are now in a position to transform $\omega^{(2)}$ and $\omega^{(3)}$:

$$\begin{aligned}\mathscr{F}^*\omega^{(2)} &= \int_0^\eta \mathbf{d}u \wedge \mathbf{d}\phi \mathrm{d}z + \mathscr{u}\mathbf{d}\eta \wedge \mathbf{d}\Phi + \tau \mathbf{d}w_1 \wedge \mathbf{d}\eta \\ &= \int_0^1 \eta \mathbf{d}u \wedge \mathbf{d}\phi \mathrm{d}\xi + \frac{1}{\eta} U|_{\xi=1} \mathbf{d}\eta \wedge \mathbf{d}\Psi + \tau \mathbf{d}w_1 \wedge \mathbf{d}\eta \\ &= \int_0^1 \mathbf{d}U \wedge \mathbf{d}\psi \mathrm{d}\xi + \mathbf{d}W_1 \wedge \mathbf{d}\eta\end{aligned}$$

and similarly

$$\begin{aligned}\mathscr{F}^*\omega^{(3)} &= \int_0^\eta \mathbf{d}v \wedge \mathbf{d}\phi \mathrm{d}z + \mathscr{v}\mathbf{d}\eta \wedge \mathbf{d}\Phi + \tau \mathbf{d}w_2 \wedge \mathbf{d}\eta \\ &= \int_0^1 \mathbf{d}V \wedge \mathbf{d}\psi \mathrm{d}\xi + \mathbf{d}W_2 \wedge \mathbf{d}\eta.\end{aligned}$$

In summary the transformed Hamiltonian system on the constant multi-symplectic structure is as follows. The Hamiltonian functional $\tilde{S}(\tilde{\boldsymbol{Z}})$ is given in (A 3), the coordinates $\tilde{Z}$ are defined in (A 2) and the three symplectic forms are

$$\begin{aligned}\tilde{\omega}^{(1)} &= \mathscr{F}^*\omega^{(1)} = -\mathrm{d}\Psi \wedge \mathrm{d}\eta, \\ \tilde{\omega}^{(2)} &= \mathscr{F}^*\omega^{(2)} = \int_0^1 \mathrm{d}U \wedge \mathrm{d}\psi \mathrm{d}\xi + \mathrm{d}W_1 \wedge \mathrm{d}\eta, \\ \tilde{\omega}^{(3)} &= \mathscr{F}^*\omega^{(3)} = \int_0^1 \mathrm{d}V \wedge \mathrm{d}\psi \mathrm{d}\xi + \mathrm{d}W_2 \wedge \mathrm{d}\eta.\end{aligned}$$

The gradient of $\tilde{S}$ and the transformed symplectic operators $\tilde{\mathbf{M}}$, $\tilde{\mathbf{K}}$ and $\tilde{\mathbf{L}}$ are defined with respect to the following inner product, which does not depend on $\tilde{\boldsymbol{Z}}$:

$$\langle \boldsymbol{F}, \boldsymbol{G} \rangle = F_1 G_1 + G_2 G_2 + F_3 G_3 + F_4 G_4 + \int_0^1 [F_5 G_5 + F_6 G_6 + F_7 G_7] \mathrm{d}\xi$$

from which we find

$$\tilde{\omega}^{(1)}(\boldsymbol{F}, \boldsymbol{G}) = \langle \tilde{\mathbf{M}} \boldsymbol{F}, \boldsymbol{G} \rangle, \quad \tilde{\omega}^{(2)}(\boldsymbol{F}, \boldsymbol{G}) = \langle \tilde{\mathbf{K}} \boldsymbol{F}, \boldsymbol{G} \rangle, \quad \tilde{\omega}^{(3)}(\boldsymbol{F}, \boldsymbol{G}) = \langle \tilde{\mathbf{L}} \boldsymbol{F}, \boldsymbol{G} \rangle,$$

with

$$\tilde{\mathbf{M}} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{\mathbf{K}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\mathbf{L}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

and so the governing equations take the form

$$\tilde{\mathbf{M}}\tilde{Z}_t + \tilde{\mathbf{K}}\tilde{Z}_x + \tilde{\mathbf{L}}\tilde{Z}_y = \nabla\tilde{S}(\tilde{Z}), \quad \tilde{Z} \in \mathbb{X}, \tag{A 5}$$

where the phase space is now a linear space of functions defined on the interval $[0, 1]$. Equation (A 5) is now qualitatively in the form (2.1).

## REFERENCES


Ablowitz, M. J. & Segur, H. 1979 On the evolution of packets of water waves. *J. Fluid Mech.* **92**, 691–715.

Alexander, J. C., Pego, R. L. & Sachs, R. L. 1997 On the transverse instability of solitary waves in the Kadomtsev-Petviashvili equation. *Phys. Lett.* A **226**, 187–192.

Allen, M. A. & Rowlands, G. 1993 Determination of the growth rate for the linearized Zakharov-Kuznetsov equation. *J. Plasma Phys.* **50**, 413–424.

Allen, M. A. & Rowlands, G. 1995 Stability of obliquely propagating plane solitons of the Zakharov-Kuznetsov equation. *J. Plasma Phys.* **53**, 63–73.

Allen, M. A. & Rowlands, G. 1997 On the transverse instability of solitary waves. *Phys. Lett.* A **235**, 145–146.

Benney, D. J. & Luke, J. C. 1984 Interactions of permanent waves of finite amplitude. *J. Math. Phys.* **43**, 309–313.

Bettinson, D. C. & Rowlands, G. 1997 Transverse stability of plane solitons using the variational method. *J. Plasma Phys.* **57**, 111–124.

Bridges, T. J. 1996 Periodic patterns, linear instability, symplectic structure and mean-flow dynamics for three-dimensional surface waves. *Phil. Trans. R. Soc. Lond.* A **354**, 533–574.

Bridges, T. J. 1997*a* Multi-symplectic structures and wave propagation. *Math. Proc. Camb. Phil. Soc.* **121**, 147–190.

Bridges, T. J. 1997*b* A geometric formulation of the conservation of wave action and its implications for signature and the classification of instabilities. *Proc. R. Soc. Lond.* A **453**, 1365–1395.

Bridges, T. J. 1998 Toral-equivariant partial differential equations and quasiperiodic patterns. *Nonlinearity* **11**, 467–500.

Bridges, T. J. 1999 A new framework for studying the stability of genus-1 and genus-2 KP patterns. *Eur. J. Mech.* B/*Fluids* **18**, 493–500.

Bridges, T. J. 2000 Universal geometric condition for the transverse instability of solitary waves. *Phys. Rev. Lett.* **84**, 2614–2617.

Bridges, T. J. & Derks, G. 1999 Unstable eigenvalues, and the linearization about solitary waves and fronts with symmetry. *Proc. R. Soc. Lond.* A **455**, 2427–2469.

Bridges, T. J. & Derks, G. 2000 Linear instability of solitary wave solutions of the Kawahara equation and its generalizations. UniS Preprint: `http://www.maths.surrey.ac.uk/`.

Gottwald, G., Grimshaw, R. H. J. & Malomed, B. A. 1998 Stable two-dimensional parametric solitons in fluid systems. *Phys. Lett.* A **248**, 208–218.

Grillakis, M., Shatah, J. & Strauss, W. 1990 Stability theory of solitary waves in the presence of symmetry, II. *J. Funct. Anal.* **94**, 308–348.

Hărăguş-Courcelle, M. & Ill'ichev, A. 1998 Three-dimensional solitary waves in the presence of additional surface effects. *Eur. J. Mech.* B/*Fluids* **17**, 739–768.

Infeld, E. & Rowlands, G. 1990 *Nonlinear Waves, Solitons and Chaos*. Cambridge University Press.

Infeld, E., Rowlands, G. & Senatorski, A. 1999 Instabilities and oscillations of one- and two-dimensional Kadomtsev-Petviashvili waves and solitons. *Proc. R. Soc. Lond.* A **455**, 4363–4381.

IWASAKI, H., TOH, S. & KAWAHARA, T. 1990 Cylindrical quasi-solitons of the Zakharov-Kuznetsov equation. *Physica* D **43**, 293–303.

JANSSEN, P. A. E. M. & RASMUSSEN, J. J. 1983 Nonlinear evolution of the transverse instability of plane-envelope solitons. *Phys. Fluids* **26**, 1279–1287.

JOHNSON, R. S. 1996 A two-dimensional Boussinesq equation for waves and some of its solutions. *J. Fluid Mech.* **323**, 65–78.

KADOMTSEV, B. B. & PETVIASHVILI, V. I. 1970 On the stability of solitary waves in weakly dispersing media. *Sov. Phys. Dokl.* **15**, 539–541.

KIVSHAR, Y. S. & PELINOVSKY, D. E. 2000 Self-focusing and transverse instabilities of solitary waves. *Phys. Rep.* **331**, 117–195.

KUZNETSOV, E. A., RUBENCHIK, A. M. & ZAKHAROV, V. E. 1986 Soliton instability in plasmas and hydrodynamics. *Phys. Rep.* **142**, 104–165.

LAEDKE, E. W. & SPATSCHEK, K. H. 1978 Nonlinear stability of envelope solitons. *Phys. Rev. Lett.* **41**, 1798–1801.

LONGUET-HIGGINS, M. S. 1974 On the mass, momentum, energy and circulation of a solitary wave. *Proc. R. Soc. Lond.* A **337**, 1–37.

LONGUET-HIGGINS, M. S. 1978 The instability of gravity waves of finite amplitude in deep water. I. Superharmonics. *Proc. R. Soc. Lond.* A **360**, 471–488.

LONGUET-HIGGINS, M. S. & TANAKA, M. 1997 On the crest instabilities of steep surface waves. *J. Fluid Mech.* **336**, 51–68.

MAKHANKOV, V. G. 1978 Dynamics of classical solitons in non-integrable systems. *Phys. Rep.* **35**, 1–128.

OSTROVSKY, L. & SHRIRA, V. 1976 Instability and self-refraction of solitons. *Sov. Phys. JETP* **44**, 738–743.

PELINOVSKY, D. E. & GRIMSHAW, R. H. J. 1996 An asymptotic approach to solitary-wave instability and critical collapse in long-wave KdV-type evolution equations. *Physica* D **98**, 139–155.

PELINOVSKY, D. E. & GRIMSHAW, R. H. J. 1997 Asymptotic methods in soliton stability theory. In *Advances in Fluid Mechanics Series*, vol. 12: *Nonlinear Instability Analysis* (ed. L. Debnath & S. R. Choudhury), pp. 245–312. Computational Mechanics Publications, Southampton.

PEGO, R. L. & QUINTERO, J. R. 1999 Two-dimensional solitary waves for a Benney-Luke equation. *Physica* D **132**, 476–496.

RYPDAL, K. & RASMUSSEN, J. J. 1989 Stability of solitary structures in the nonlinear Schrödinger equation. *Physica Scripta* **40**, 192–201.

SAFFMAN, P. G. 1985 The superharmonic instability of finite-amplitude water waves. *J. Fluid Mech.* **159**, 169–174.

SHRIRA, V. 1980 Nonlinear refraction of solitons. *Sov. Phys. JETP* **52**, 44–49.

SHRIRA, V. & PESENSON, M. Z. 1984 Soliton stability with respect to transverse perturbations of finite length and amplitude. In *Nonlinear and Turbulent Processes in Physics* (ed. R. Z. Sagdeev), pp. 1061–1068. Gordon & Breach; Harwood.

SKRYABIN, D. 2000 Stability of multi-parameter solitons: Asymptotic approach. *Physica* D **139**, 186–193.

SPATSCHEK, K. H., SHUKLA, P. K. & YU, M. Y. 1975 On the two-dimensional stability of ion-acoustic solitons. *Phys. Lett.* A **54**, 419–420.

TANAKA, M. 1985 The stability of steep gravity waves. Part 2. *J. Fluid Mech.* **156**, 281–289.

TANAKA, M. 1986 The stability of solitary waves. *Phys. Fluids* **29**, 650–655.

TANAKA, M., DOLD, J. W., LEWY, M. & PEREGRINE, D. H. 1987 Instability and breaking of a solitary wave. *J. Fluid Mech.* **185**, 235–248.

ZAKHAROV, V. E. 1975 *JETP Lett.* **22**, 172–173.

ZAKHAROV, V. E. & KUZNETSOV, E. A. 1974 Three-dimensional solitons. *Soviet Phys. JETP* **39**, 285–286.